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ON FUNCTIONS OF STRENGTH t

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For a finite set X, a function $f\colon \mathbf{P}(X)\to\mathbf{Z}$ is said to have strength t if $\sum_{A\subseteq B} f(B)=0$ for all $A\in\mathbf{P}(X)$, $|A|\leqq t$. Supports of functions of strength t define a matroid on $\mathbf{P}(X)$. We study the circuits in this matroid. Some other related problems are also discussed.

1. Introduction and statements of results

Let $X=\{x_1, x_2, ..., x_n\}$ be a finite set of n elements. We will denote by $\mathbf{P}(X)$, the set of all subsets of X and by $\mathbf{P}_k(X)$, the set of all k-subsets of X. We will denote by $V=V(X)$, the free $\mathbf{Z}$-module of all integral functions $f\colon \mathbf{P}(X)\to\mathbf{Z}$.

For every $B\subseteqq X$ and integer $t\geqq 0$, let $\chi_B^t\in V$ be a function defined by

$$\chi_B^t(C)=\begin{cases}1 & \text{if } B\supseteqq C \text{ and } |C|\leqq t\\ 0 & \text{otherwise.}\end{cases}$$

Define a subset $S\subseteqq\mathbf{P}(X)$ to be *independent* if and only if $\{\chi_B^t | B\in S\}$ is a linearly independent subset of V. This clearly induces a matroid structure on $\mathbf{P}(X)$. This matroid has been studied in connection with many problems in the theory of designs or extremal problems for families of finite sets. In fact many interesting inequalities are proved by showing that certain families of subsets of X correspond to independent sets in this matroid (see [1, 3, 4, 5, 6, 7, 12]).

The aim of this paper is to study circuits in this matroid. In this section we first redefine the matroid described above in 'dual' terms which seem more suitable and familiar and then describe various results. The next section is devoted to the proofs.

For any subset $A\subseteqq X$, define a monomial $p(A)=\prod_{x\in A} x$, $P(\emptyset)=1$. We identify A with $p(A)$ and in this way we also think of $\mathbf{P}(X)$ as the set of all square-free monomials in $x_1, x_2, ..., x_n$. Similarly we also consider any $f\in V(X)$ as a polynomial $f=\sum_{A\subseteqq X} f(A)p(A)$. Thus $V(X)$ is also free module of all square-free polyno-

AMS subject classification (1980): 05 B 35 mials in $x_1, x_2, \ldots, x_n$ with integral coefficients. Clearly $V(X)$ is free module with $\mathbf{P}(X)$ as a basis.

For every $f \in V$, let us denote by ∂f the unique element of V defined by $\partial f(B) = \sum_{B \subseteq A \subseteq X} f(A)$ for all $B \subseteq X$.

A function $f \in V$ is said to be of *strength* t iff $\partial f(B) = 0$ for all $B \subseteq X$ with $|B| \leqq t$.

Functions of strength t have been studied earlier and called null-t-designs in [8, 11], 0-measures, 0-configurations in [2, 3]. We give two examples of such function (see [3, 8] for details).

Example 1(a). For any sequence $y = (y_1, y_2, \ldots, y_{k+t+1})$, $y_i \in X$, $k \geqq t+1$ and $y_i \neq y_j$ for $i \neq j$, the polynomial $f_y = (y_1 - y_2)(y_3 - y_4) \ldots (y_{2t+1} - y_{2t+2}) y_{2t+3} \ldots y_{k+t+1}$ is a function of strength t.

Note that $f_y(B) = 0$ if $|B| \neq k$.

Example 1(b). For any sequence $y = (y_1, y_2, \ldots, y_{t+1})$, $y_i \in X$, $y_i \neq y_j$ for $i \neq j$, the polynomial

$$g_y = \prod_{1 \leqq i \leqq t+1} (1 - y_i)$$

is a function of strength t.

Define $V_t = V_t(X) = \{f \mid f \in V,\ f \text{ has strength } t\}$. V_t is clearly a submodule of V and will be called *module of functions of strength t over X*.

For a function $f \in V$ define

(i) $S(f) = \text{support}(f) = \{B \in \mathbf{P}(X) \mid f(B) \neq 0\}$

(ii) $S^+(f) = \text{positive support}(f) = \{B \in \mathbf{P}(X) \mid f(B) > 0\}$

(iii) $S^-(f) = \text{negative support}(f) = \{B \in \mathbf{P}(X) \mid f(B) < 0\}$.

Remark.

(a) $S^+(f) \cup S^-(f) = S(f)$, $S^+(f) \cap S^-(f) = 0$ and $S^+(-f) = S^-(f)$

(b) $S^+(f) = S(f^+)$ and $S^-(f) = S(f^-)$

where $f = f^+ - f^-$, f^+ and f^- are defined by

$$f^+(B) = \begin{cases} f(B) & \text{if } f(B) \geqq 0 \\ 0 & \text{otherwise;} \end{cases}$$

$$f^-(B) = \begin{cases} -f(B) & \text{if } f(B) \leqq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A subset $S \subseteq \mathbf{P}(X)$ is said to be

(a) *t-dependent* if and only if $S \supseteq S(f)$ for some function f of strength t, $f \neq 0$,

(b) *t-independent* if and only if S is not t-dependent,

(c) *t-circuit of X* if and only if S is t-dependent and every $S' \subsetneqq S$, is t-independent.

We will denote by $M_t(X)$, the set of all t-circuits of X.

Remark. Clearly, t-circuits of X are precisely the circuits of the matroid described in the beginning of this paper. It is also clear that this matroid is the chain-matroid corresponding to the chain group V_t in the language of Tutte [14].

We will also study minimal positive supports (defined below) of functions of strength t. A subset $S \subseteqq \mathbf{P}(X)$ will be called *t-minimal positive support* if and only if $S = S(f^+)$ for some function f of strength t and for every $S' \subsetneqq S$, $S' \neq S(g^+)$ for any function g of strength t.

The following theorem shows that the functions f_y and g_y defined in Example 1(a) and (b) have minimum support and minimum positive support.

Theorem 1. *Let f be a function of strength t, $f \not\equiv 0$, then*

(a) $S(f) \geqq 2^{t+1}$

(b) $S^+(f) \geqq 2^t$.

The statement (a) of Theorem 1 has been proved in [6]. One can also easily see that the method of proof of (a) in [6] also yields statement (b) of Theorem 1.

Remark. We also note that proof of Theorem 1 shows that Theorem 1(a) is also true for functions of strength t, taking values in any ring in place of $\mathbf{Z}$.

Corollary 1. *If $S \subseteqq \mathbf{P}(X)$ and $|S| < 2^{t+1}$ then $\{\chi_B | B \in S\}$ is a linearly independent subset of V.*

The following theorem shows that every 1-minimal positive support has cardinality 2.

Theorem 2. *Let S be a nonzero function of strength 1, with 1-minimal positive support then $S^+(f) = 2$.*

The following examples show that for $t > 1$ there are t-circuits (t-minimal positive supports) of cardinality more than 2^{t+1} (respectively 2^t).

Example 2(a).

Let $f_1 = (1 + y_1 z_2 + y_2 z_1 + x_1 y_2 + x_2 y_1 + x_1 z_2 + x_2 z_1 + x_1 y_1 z_1 + x_2 y_2 z_2)$
$- (z_1 + x_1 + y_1 + x_1 y_2 z_1 + x_1 z_2 y_1 + y_1 x_2 z_1 + x_2 y_2 + x_2 z_2 + y_2 z_2)$,
and $f_2 = (1 + x_1 y_1 z_1 + x_1 y_2 + x_2 y_1 + y_2 z_1 + y_1 z_2 + z_1 x_2 + z_2 x_1)$
$- (x_1 y_2 z_1 + x_1 z_2 y_1 + y_1 x_2 z_1 + x_1 + y_1 + z_1 + x_2 + y_2 + z_2)$.

It can be easily verified that $f_1, f_2 \in V_2(X)$ for $X = \{x_1, y_1, z_1, x_2, y_2, z_2\}$. Using Theorem 3 described below it can also be easily verified that $S(f_1)$ is a 2-circuit for X.

Example 2(b).

Let $f = (2 + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4) - (x_1 + x_2 + x_3 + x_4 + 2 x_1 x_2 x_3 x_4)$.

Again using Theorem 3 and 4 described below one can easily verify that $S(f)$ is a 2-circuit on $X = \{x_1, x_2, x_3, x_4\}$ and $S^+(f)$ is a 2-minimal positive support.

Remark. From Example 2(a), since $S^+(f_2) \subsetneqq S^+(f_1)$, it also follows that in general positive support of a function $f \in V_t$, for which $S(f)$ is a t-circuit, need not be a t-minimal positive support.

A good description of all t-circuits will indeed help very much in many interesting problems in the theory of designs. However it seems to be a difficult problem.

The following two theorems describe criteria to determine whether for a given function $f\in V_t$, $S(f)$ is t-circuit or $S^+(f)$ is a t-minimal positive support.

Theorem 3. *Let $S\subseteqq \mathbf{P}(X)$, then S is t-circuit if and only if the following conditions are satisfied.*
(i) *There exists $f\in V_t$ such that $S(f)=S$,*
(ii) *The rank of the submodule of V generated by $\{\chi_B^t|B\in S\}$ is $|S|-1$.*

Theorem 4. *Let $S\subseteqq \mathbf{P}(X)$, then S is a t-minimal positive support if and only if the following conditions are satisfied.*
(i) *There exists $f\in V_t$ such that $S^+(f)=S$.*
(ii) *For any $g\in V$ satisfying*
(a) $g(B)\geqq 0$ *for all* $B\in \mathbf{P}(X)$
(b) $f^+-g\in V_t$
the following condition holds
(c) *for every* $B\in S^+(f)\cap S(g)$, $f(B)>g(B)$.

The following theorem shows that Theorem 1 can be strengthened in special cases. Note that Example 2(b) shows that it will be difficult to strengthen Theorem 1 in general.

Theorem 5. *If $f\in V_t(X)$ with $S(f)\subseteqq \mathbf{P}_{(t+1)}(X)$, then either*
(i) $f=cf_y$ *for some vector* $y=(y_1, y_2, y_3, \dots, y_{2t+2})$, *and* $c\in \mathbf{Z}$, *where* f_y *is as described in Example 1(a) or*
(ii) $|S(f)|\geqq \frac{3}{2}\cdot 2^{t+1}$ and $S^+(f)\geqq \frac{3}{2}\cdot 2^t$.

We will also prove a few simple results on the structure of $S(f), f\in V_t$ in general.

Theorem 6. *Let $f\in V_t(X)$ then $S(f)$ is the union of all t-circuits C, $C\in S(f)$.*

Theorem 6 is in fact true in general for any chain matroid (see [14]). The following theorem gives an analogue of Fisher type equations for t-designs in the case of functions of strength t.

Theorem 7. *(Fisher-type equations.) Let $f\in V_t(X)$, then for all $0\leqq i\leqq t$.*

$$\sum_{C\in \mathbf{P}(X)} \binom{|B\cap C|}{i} f(C) = 0 \quad \textit{for all} \quad B\in S(f).$$

The following corollary is immediate.

Corollary 2. *Let $f\in V_t(X)$, then for any polynomial F of degree $\leqq t$ with integral coefficients*

$$\sum_{C\in \mathbf{P}(X)} F(|B\cap C|)f(C) = 0 \quad \textit{for all} \quad B\in S(f).$$

The above corollary will be used to prove the following theorem which generalises Theorem 6 in [3], Graham [9] has informed us that this theorem has also been proved by Chung and Graham however their proof is different from the one given here.

Theorem 8. *Let $f \in V_t(X)$ and p be any prime number. Let $B \in S(f)$ be such that $f(B) \not\equiv 0 \pmod p$, then either*
(i) $|B \cap C| \not\equiv |B| \pmod p$ *for some* $C \in S(f)$, $C \neq B$, *or*
(ii) *there exist* $C_1, C_2, \ldots, C_{t+1} \in S(f)$, $C_i \neq B$, *such that for all* $1 \leqq i \neq j \leqq t+1$, $|B \cap C_i| \neq |B \cap C_j| \pmod p$.

The following corollary is immediate.

Corollary 3. *Let $f \in V_t(X)$ then for every $B \in S(f)$ there exist $C_1, C_2, \ldots, C_{t+1} \in S(f)$, $C_i \neq B$, satisfying for all $1 \leqq i \neq j \leqq t+1$,*

$$|B \cap C_i| \neq |B \cap C_j|.$$

The following result which follows from Theorem 8, was first proved in [4].

Corollary 4. *Let $S \subseteqq \mathbf{P}_k(X)$, and p be a prime number. Suppose there exist integers $\mu_1, \mu_2, \ldots, \mu_t$, $\mu_i \not\equiv k \pmod p$, such that for all $B, C \in S$, $B \neq C$, $|B \cap C| \equiv \mu_i \pmod p$ for some i, then*

$$|S| \leqq \binom{v}{t}.$$

Finally we state various problems which come to one's mind in the context of studying functions of strength t, and discuss a few related results.

Problem 1. *Give a good combinatorial description of*
(a) *all t-circuits,*
(b) *all t-minimal positive sets,*
(c) *positive supports of functions of strength t.*

For any $f \in V$ define a function $\partial_t f \in V$ by

$$\partial_t f(B) = \begin{cases} \partial f(B) & \text{if } |B| = t \\ 0 & \text{otherwise.} \end{cases}$$

Problem 1(c) can clearly be reformulated as follows.

Problem 1(c′). *Describe all sets $S \subseteqq \mathbf{P}(X)$ such that there exists a function $f \in V$ satisfying the following conditions*
(i) $f(B) \geqq 0$ *for all* $B \in \mathbf{P}(X)$, $S(f) = S$
(ii) *there exists a function $g \in V$, such that*
I. $S(g) \cap S = \emptyset$, $g(B) \geqq 0$ *for all* $B \in \mathbf{P}(X)$.
II. $\partial_i g = \partial_i f$ *for all* $0 \leqq i \leqq t$.
Thus this problem is related to the following.

Problem 2. *Describe the set $E_t(X)$ of all ordered $(t+1)$-tuples $g = (g_0, g_1, \ldots, g_t)$, $g_i \in V(X)$, such that there exists a function $f \in V(X)$ satisfying*
(a) $f(B) \geqq 0$ *for all* $B \in \mathbf{P}(X)$,
(b) $\partial_i f = g_i$ $0 \leqq i \leqq t$.

Problem 3. *Describe the set $E_t^k(X) \subseteqq E_t(X)$ of all $g \in E_t(X)$, for which there exists $f \in V(X)$ satisfying* (a) *and* (b) *above with $S(f) \subseteqq \mathbf{P}_k(X)$.*

The problems described above are quite difficult in their full generality. The well known problem on existence of $t-(v, k, \lambda)$-designs, for example, corresponds to describing all ordered tuples $(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_t)$ of integers for which $(\lambda_0 e_0, \lambda_1 e_1, \dots \dots, \lambda_t e_t) \in E_t^k(X)$, where $e_i \in V$ is defined by

$$e_i(B) = \begin{cases} 1 & \text{if } |B| = i \\ 0 & \text{otherwise.} \end{cases}$$

We also remark here that Problems 2 and 3 can be solved if we do not insist for function f to satisfy the condition (a) viz. $f(B) \geqq 0$. In fact this question has been considered in [10, 11] and [13] for the case $g=(\lambda_1 e_1, \lambda_2 e_2, \dots, \lambda_t e_t)$. However the proofs given there are also valid for general g. We state four theorems below for general g. The first one is trivial. We do not give proofs for the others as the proofs are essentially along the same lines as that of the main theorem in [11] and Theorem 1 and 3 in [13].

Theorem 9. *Let $g=(g_0, g_1, \dots, g_t)$ be an ordered tuple, $g_i \in V$ such that $S(g_i) \subseteqq \mathbf{P}_i(X)$, then there exists $f \in V$ s.t. $\partial_i f = g_i$ for all $0 \leqq i \leqq t$.*

Theorem 10. *Let $g=(g_0, g_1, \dots, g_t)$ be an ordered tuple $g_i \in V$ for which there exists a positive integer μ and some $f \in V$ satisfying*
(a) *$f(B) \geqq 0$ for all $B \in \mathbf{P}(X)$ and $f(B)=0$ for all $B \in \mathbf{P}_i(X)$, $0 \leqq i \leqq t$,*
(b) *$\partial_i f = \mu g_i$ for all $0 \leqq i \leqq t$,*
then there exists an integer λ_g such that $\lambda g \in E_t(X)$ for all $\lambda > \lambda_g$.

Theorem 11. *Let $g=(g_0, g_1, g_2, \dots, g_t)$ be an ordered tuple such that $S(g_i) \subseteqq \mathbf{P}_i(X)$ then the following condition* (N) *is necessary and sufficient for the existence of a function $f \in V$, satisfying $\partial_i f = g_i$ for $0 \leqq i \leqq t$ and $S(f) \subseteqq \mathbf{P}_k(X)$:*

(N) $$\partial_i g_{i+1} = (k-i) g_i, \quad 0 \leqq i \leqq t-1.$$

Theorem 12. *Let $g=(g_0, g_1, g_2, \dots, g_t)$ be an ordered tuple, $g_i \in V$, for which there exists a positive integer μ and some $f \in V$ satisfying*

(a) *$f(B) \geqq 0$ for $B \in \mathbf{P}(X)$ and $S(f) = \mathbf{P}_k(X)$*

(b) $\partial_i(f) = \mu g_i$ for all $0 \leqq i \leqq t$

then there exists an integer λ_g such that $\lambda g \in E_t^k(X)$ for all $\lambda > \lambda_g$.

Note that condition (b) above already implies that the necessary condition (N) is satisfied by g.

2. Proofs

Proof of Theorem 2. Let us suppose that $f \in V_1$ is such that $S^+(f)$ is 1-minimal positive support; we will show that $|S^+(f)|=2$. Let $S=S^+(f)$ and let $S = \{B_1, B_2, \dots, B_m\}$. Now if there exist B and B' in S such that $B \not\subset B'$ or $B \not\supset B'$, then the function $g \in V$, defined by

$$g(B) = g(B') = 1, \quad g(B \cap B') = g(B \cup B') = -1, \quad g(C)=0$$

otherwise is nonzero and has strength 1.

Thus by the minimality of S we have either $m=2$ or S is a chain, i.e., we can assume that $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_m$.

Again if $|B_m - B_1| \geqq 2$, we can take two subsets B and B' of X, different from B_1 and B_m, such that $B \cap B' = B_1$ and $B \cup B' = B_m$ and hence the function $-g \in V_1$ for g as defined above shows that $S = \{B_1, B_m\}$. Thus, in any case $|S| \leqq 2$. ∎

Proof of Theorem 3. Let S be a t-circuit and let $g, f \in V_t(X)$ be such that $S(g) = S(f) = S$. Now if $g \neq rf$ for a rational number r, we can clearly find integers m_1 and m_2 such that $S(m_1 f + m_2 g) \subsetneqq S$ a contradiction as S is a t-circuit. Thus for every $h \in V_t(X)$ with $S(h) = S$ there exists a rational number r with $h = rf$. This is clearly equivalent to saying that the coefficient matrix of the equations (2.1) in variables y_B, $B \in S$, described below has rank $|S|-1$.

$$(2.1) \qquad P_A\colon \sum_{A \subseteq B \in S} y_B = 0 \quad \text{for all} \quad A \in \mathbf{P}_i(X), \quad 0 \leqq i \leqq t.$$

This is equivalent to saying that the rank of the submodule of V generated by $\{\chi_B^t | B \in S\}$ is $|S|-1$. ∎

Proof of Theorem 4. If condition (c) is not satisfied then $S^+(f^+ - g) \subsetneqq S$, which contradicts the fact that S is a t-minimal positive support. The converse is also clear. ∎

Proof of Theorem 5. We will prove the theorem by induction. The assertion can be easily verified for $t=0$ and 1. Now let $t>1$, and let T be defined by

$$T = \bigcup_{B \in S(f)} B.$$

Now $|T| \geqq 2t+2$ and if $|T| = 2t+2$ then (i) holds (see [6]).

Hence we may assume that $|T| \geqq 2t+2$. We will prove the theorem by induction on $|T|$. For $x \in T$, define $f_x \in V$ by

$$f_x(B) = \begin{cases} 0 & \text{if} \quad x \in B \subseteq X \\ f(B \cup \{x\}) & \text{if} \quad x \notin B \subset X. \end{cases}$$

Clearly, $f_x \in V_{t-1}(X)$ and

$$S(f_x) = \{B - \{x\} | x \in B \in S(f)\}.$$

Now we consider two cases.

Case 1. For some $x \in T$, (i) holds for f_x, i.e.

$$f_x = c(y_1 - y_2) \ldots (y_{2t-1} - y_{2t})$$

for some $y_i \in T$.

Since $|T| \geqq 2t+2$ we can choose $y_{2t+1} \in T$ different from $y_i (1 \leqq i \leqq 2t)$ and x, consider

$$f' = f + (y_{2t+1} - x) f_x.$$

Now $f' \in V_t(X)$, and $\bigcup_{B \in S(f')} B \subseteq T - \{x\}$. Hence using the induction hypothesis for f', we have either

$$\text{(a)} \qquad |S(f')| \geqq \frac{3}{2} \cdot 2^{t+1} \quad \text{and} \quad |S^+(f')| \geqq \frac{3}{2} \cdot 2^t,$$

in which case, using the fact that $S(xf_x) \subseteqq S(f) - S(f')$, it can easily be checked that $|S(f)| \geqq \frac{3}{2} \cdot 2^{t+1}$ and $|S^+(f)| \geqq \frac{3}{2} \cdot 2^t$; or we have

(b) $$f' = c'(x_1' - x_2') \cdot \ldots \cdot (x_{2t+1}' - x_{2t+2}') \quad \text{for some} \quad x_i' \in X.$$

Then again the theorem can be checked directly, for

$$f = c'(x_1' - x_2') \ldots (x_{2t+1}' - x_{2t+2}') + c(y_{2t+1} - x)(y_1 - y_2) \ldots (y_{2t-1} - y_{2t}).$$

Case 2. For all $x \in T$ (ii) holds with f_x, i.e.

$$|S(f_x)| \geqq \frac{3}{2} \cdot 2^t \quad \text{and} \quad |S^+(f_x)| \geqq \frac{3}{2} \cdot 2^{t-1}.$$

Now (ii) follows from $|T| \geqq 2t+2$ and

$$|S(f)| = \frac{1}{t+1} \sum_{x \in T} |S(f_x)|.$$

$$|S^+(f)| = \frac{1}{t+1} \sum_{x \in T} |S^+(f_x)|$$

This completes the proof. ∎

Proof of Theorem 6. Theorem 6 follows easily by induction on $|S(f)|$. ∎

Proof of Theorem 7.

$$\sum_{C \in \mathbf{P}(X)} \binom{|B \cap C|}{i} f(C) = \sum_{\substack{A \subseteqq B \\ |A| = i}} \Big(\sum_{A \subseteqq C} f(C) \Big) = \sum_{\substack{A \subseteqq B \\ |A| = i}} \partial f(A) = 0. \quad ∎$$

Proof of Theorem 8. Let $B \in S(f)$ be such that condition (i) is not satisfied. Now if (ii) is not satisfied there exist integers $\mu_1, \mu_2, \ldots, \mu_t$; $\mu_i \not\equiv |B| \pmod p$ such that for all $C \in S(f)$, there exists i, $1 \leqq i \leqq t$ satisfying

(2.2) $$|B \cap C| \equiv \mu_i \pmod p.$$

Define a polynomial $F(t)$ by

$$F(t) = \prod_{i=1}^{t} (t - \mu_i)$$

using Corollary 2 for $F(t)$ and (2.2) above we get

$$f(B) \prod_{i=1}^{t} (|B| - \mu_i) \equiv 0 \pmod p.$$

Since p is a prime and $|B| \not\equiv \mu_i \pmod p$ we infer $f(B) \equiv 0 \pmod p$, contradicting the assumptions. This completes the proof. ∎

Proof of Corollary 4. Using Theorem 8, we obtain that S is t-independent, i.e. $\{\chi_B | B \in S\}$ is linearly independent, thus $|S| \leqq \binom{v}{t}$.

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